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THE SUFFICIENT CONDITIONS OF STABILITY FOR THE VALUE FUNCTION OF A DIFFERENTIAL GAME IN TERMS OF SINGULAR POINTS†

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Differential games in which the payoff is the time required to reach a given terminal set (optimum-time games) are considered. The Bellman-Isaacs equation must have a classical solution in domains where the value function of the game is smooth. At singular points, where smooth branches of the solution interlock, more complicated conditions must hold, in terms of directional derivatives. A class of games is found and two types of singularities are described such that the aforementioned conditions follow automatically from the geometrical properties of the singularities: the class of games with autonomous separated dynamics in which one player's control set is a line segment. An appropriate theorem is proved. The result is used to construct the value function of a game in which there are three types of singular surfaces: dispersal, equivocal, and switching surfaces. © 2003 Elsevier Ltd. All rights reserved.

Given an optimum-time differential game, one can formulate sufficient conditions for an arbitrary function to be the value function of that game [1, 2]. If the value function of the game is smooth, it may be determined by solving a Cauchy problem for a certain first-order partial differential equation (the Bellman-Isaacs equation) [3]. Under certain additional conditions, the method of classical characteristics may be used to construct a solution of the Cauchy problem [4]. However, only in exceptionally rare cases is the value function smooth. Nevertheless, the method of characteristics may be used to construct a piecewise-smooth value function. The construction method, due to Isaacs [3], amounts to successive determination of smooth branches of the solution by the method of classical characteristics.

The main difficulty in applying Isaacs' method is to find the surfaces on which smooth branches of the solution are spliced together (singular surfaces). Various types of singular surfaces have been considered, as have some methods for constructing them [3], but on the whole there is no rigorous proof of the fact that a piecewise-smooth function obtained in this way will be the value function of a differential game. From the modern point of view, that entails verifying stability conditions on the singular surfaces of the function constructed [2.5].

The result of this paper includes a definition of certain types of singular point and a proof that stability conditions hold at such points. As an example of the application of the result, the value function of a brachistochrone game problem will be constructed.

1. THE MINIMUM-TIME GAME PROBLEM. VALUE FUNCTION AND STABILITY CONDITIONS

Let the motion of a control system be described by a differential equation

$$\dot{x}(t) = f(x(t), u(t), \upsilon(t)), \quad t \geq 0 \tag{1.1}$$

where $x(t) \in R^n$ is the phase state of the system at time t, $u(t) \in P$ and $\upsilon(t) \in Q$ are the controls of the first and second players, and $P \subset R^m$ and $Q \subset R^l$ compact sets. The function $f(x, u, \upsilon)$ is assumed to be continuous jointly in its variables and to satisfy a Lipschitz condition

$$\|f(x, u, \upsilon) - f(y, u, \upsilon)\| \leq L\|x - y\|, \quad x, y \in R^n, \quad u \in P, \quad \upsilon \in Q$$

where L is a constant. In addition, let us assume that Isaacs' condition (the saddle-point condition in the small game [1]) is satisfied

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$$H(x, p) := \min_{u \in P}\max_{v \in Q}\langle p, f(x, u, v)\rangle = \max_{v \in Q}\min_{u \in P}\langle p, f(x, u, v)\rangle, \quad x, p \in R^n \tag{1.2}$$

Positional strategies [1] for the first and second players are defined as arbitrary functions $U: R^n \to P$ and $V: R^n \to Q$. Strategies U and V generate pencils $X_1(x_0, U)$ and $X_2(x_0, V)$ of constructive motions [1, 6] that emanate from position x_0 at time $t = 0$.

A constructive motion $x(\cdot) \in X_1(x_0, U)$ is defined as a function $x(t)$ for which, in any interval $(0, \vartheta)$, there is a sequence of Euler polygonal lines $x^{(k)}(t)$ defined by the conditions

$$\dot{x}^{(k)}(t) = f(x^{(k)}(t), U(x^{(k)}(\tau_i^{(k)})), v^{(k)}(t))$$
$$t \in [\tau_i^{(k)}, \tau_{i+1}^{(k)}), \quad x^{(k)}(0) = x_0, \quad \tau_0^{(k)} = 0, \quad i = 1, 2, \ldots$$

which converge uniformly to $x(t)$ and are such that $\sup_i(\tau_{i+1}^{(k)} - \tau_i^{(k)}) \to 0$ as $k \to \infty$. Here the intervals $[\tau_i^{(k)}, \tau_{i+1}^{(k)})$ $(i = 1, 2, \ldots)$ constitute a partition of the semiaxis $t \geqslant 0$, $v_k(\cdot)$ is a measurable function with values in the set Q. The elements of the set $X_2(x_0, V)$ are defined similarly.

The object of the first player is to impel the point as quickly as possible towards a given closed terminal set $M \subset R^n$. The second player tries either to prevent an encounter with M or to maximize the time till it occurs. Thus, the functional to be optimized for the minimum-time game problem has the form

$$J(x(\cdot)) := \begin{cases} \infty, \text{ if } x(t) \notin M & \text{for any} \quad t \geq 0 \\ \min\{t \geq 0: x(t) \in M\} & \text{otherwise} \end{cases}$$

If the point $x_0 \in R^n$ is such that

$$\inf_U \sup J(X_1(x_0, U)) = \sup_V \inf J(X_2(x_0, V)) =: T^0(x_0) \tag{1.3}$$

then the number $T^0(x_0) \in [0, \infty]$ is called the *value of the game* at point x_0. Under the conditions assumed for the function $f(x, u, v)$, the value of the game exists for any $x \in R^n$ [1, 5]. The function $T^0: R^n \to [0, \infty]$ is known as the value function of the game.

Closely related to the value function are the concepts of u- and v-stable functions [1, 2]. Consider a continuous function $T: R^n \to R$ such that $M = \{x \in R^n : T(x) = 0\}$ and for any $x \in R^n$ the limit

$$\partial_\eta T(x) = \lim_{\delta \to +0} \frac{T(x + \delta\eta) - T(x)}{\delta}$$

exists.

The function $T(x)$ is said to be *u-stable* (*v-stable*) if, for any $x \in R^n \backslash M$,

$$\begin{gathered} \sup_{v \in Q} \inf\{\partial_\eta T(x) : \eta \in \operatorname{co} f(x, P, v)\} \leq -1 \\ (\inf_{u \in P} \sup\{\partial_\eta T(x) : \eta \in \operatorname{co} f(x, u, Q)\} \geq -1) \end{gathered} \tag{1.4}$$

where

$$f(x, P, v) := \{f(x, u, v) : u \in P\}, \quad f(x, u, Q) := \{f(x, u, v) : v \in Q\}$$

and $\operatorname{co}\{f\}$ denotes the convex hull of the vector f.

Conditions (1.4) are conditions for the stability of the function $T(x)$ at a point $x \in R^n \backslash M$. In a domain where the function $T(x)$ is differentiable, inequalities (1.4) become the Bellman-Isaacs equation [3]

$$H(x, DT(x)) = -1 \tag{1.5}$$

The validity of inequalities (1.4) is a necessary and sufficient condition for a function $T(x)$ to be the value function of a minimum-time differential game [2].

If no good description of the function $T(x)$ is available, direct verification of conditions (1.4) is difficult. It turns out, however, that in many cases the validity of the stability conditions at a point depends entirely on the structure of the function $T(x)$ in the neighbourhood of the point. We shall now define some types of such points and prove that the stability conditions hold there.

2. SIMPLE SINGULAR POINTS

Suppose the function $T(x)$ is given and continuous in a domain $\Omega \subset R^n$. The following definitions will carry over to the case of a function $T(x)$ ideas introduced previously [7] for the value function of a game.

Definition 1. A domain $G \subset \Omega$ is called a *regular* domain of $T(x)$ if
(1) $T \in C^2(G)$ and $H(x, DT(x)) = -1, x \in G$;
(2) $H \in C^2(G \times Y)$, where Y is some neighbourhood of the set $\{DT(x): x \in G\}$.
If x^* is a point in a regular domain, then there is a neighbourhood of any arbitrarily chosen time t^* in which a unique solution $(x(t), p(t))$ of the characteristic system

$$\dot{x} = H_p(x, p), \quad \dot{p} = -H_x(x, p) \tag{2.1}$$

of Eq. (1.5) exists that satisfies the conditions

$$x(t^*) = x^*, \quad p(t^*) = DT(x^*)$$

and is such that $p(t) = DT(x(t))$ [4]. It follows from this last equality that through any point of a regular domain there passes a unique solution $x(t)$ of the differential equation

$$\dot{x} = H_p(x, DT(x))$$

which is called a characteristic of the Bellman-Isaacs equation (1.5). Thus, a field of characteristics is defined in any regular domain.

The Isaacs condition (1.2) guarantees the existence of functions $U_T(x)$ and $V_T(x)$ such that

$$H(x, DT(x)) = \langle DT(x), f(x, U_T(x), V_T(x))\rangle$$

In a regular domain G we have

$$H_p(x, DT(x)) = f(x, U_T(x), V_T(x))$$

and the function $x \to H_p(x, DT(x))$ is of class $C^1(G)$. Consequently, positional strategies $U_T(x)$ and $V_T(x)$ defined in G generate motions along the characteristics. The function $U_T(x)$ and $V_T(x)$ are bounded and therefore have finite partial limits for any $x \in \partial G$.

Definition 2. A point $x \in \Omega$ for which a regular domain $G \subset \Omega$ containing x exists is called a *regular* point; otherwise, the point is *singular*.

Surfaces, all of whose points are singular, will be called singular surfaces.

Definition 3. A singular point $x^* \in \Omega$ is said to be *simple* if
(1) a neighbourhood $G \subset \Omega$ of x^* exists such that $G = G^+ \cup \Gamma \cup G^-$, where Γ is a smooth hypersurface and $G^\pm$ are regular domains;
(2) the function $DT(x), x \in G^\pm$ has a continuous extension to the hypersurface Γ.
Let $\Sigma(T)$ be the set of simple singular points of the function $T(x)$. We shall use the following notation for a point $x^* \in \Sigma(T)$: $G^\pm$ are the regular domains (in the definition of a simple singular point) separated by the hypersurface Γ; $T^\pm(x)$ are the restrictions of the function $T(x)$ to the domains $G^\pm$; $U_T^\pm(x)$, $V_T^\pm(x)$ are positional strategies of the players, generating motions along the characteristics in the domains $G^\pm$. In addition, let

$$p^\pm := \lim DT^\pm(x), \quad u^\pm := \lim U_T^\pm(x), \quad v^\pm := \lim V_T^\pm(x)$$

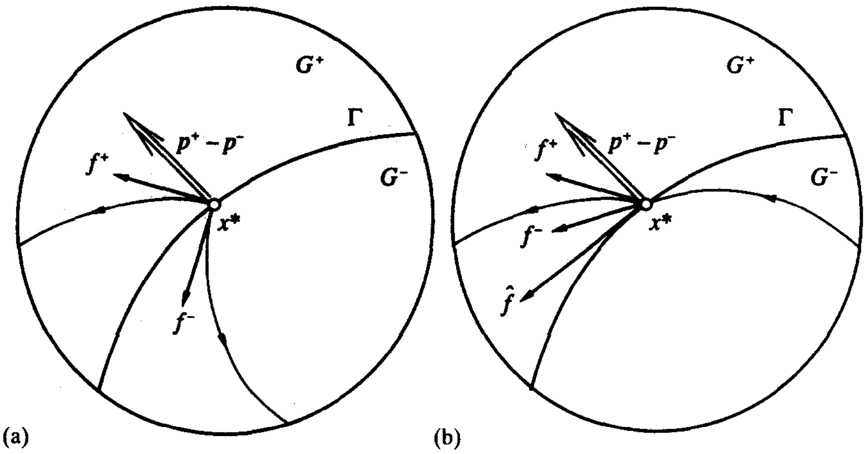
as $x \to x^*, x \in G^\pm$, and

$$f^\pm := f(x^*, u^\pm, v^\pm)$$

The symbols $\lim U_T^\pm(x)$, $\lim V_T^\pm(x)$ will denote arbitrary partial limits of the functions $U_T^\pm(x)$, $V_T^\pm(x)$ as $x \to x^*, x \in G^\pm$.

Since the function $H(x, p)$ is continuous, we have

$$\langle p^\pm, f^\pm\rangle = H(x^*, p^\pm) = -1$$



(a) (b)
Fig. 1

We note a few properties of simple singular points.

If $p^+ = p^-$, then the function $T(x)$ is differentiable at the point x^* (but not necessarily in a neighbourhood of that point), and $H(x^*, DT(x^*)) = -1$.

In the case when $p^+ \neq p^-$, the functions $T^\pm(x)$ defined on $G^\pm$ can be extended as continuous functions [7] to the entire domain G in such a way that $T^\pm \in C^1(G)$. When that is done the hypersurface Γ separating the domains $G^\pm$ may be represented as in the form

$$\Gamma = \{x \in G: T^+(x) = T^-(x)\}$$

Since the hypersurface Γ may be considered as a level surface of the function $T^+(x) - T^-(x)$, it follows that the vector $p^+ - p^-$ is orthogonal to Γ at the point x^*. In addition, the following representation is true in G

$$T(x) = \max\{T^+(x), T^-(x)\} \quad (\min\{T^+(x), T^-(x)\})$$

if the vector $p^+ - p^-$ is directed from G^- to G^+ (from G^+ to G^-). Consequently, for any vector $\eta \in R^n$ a derivative $\partial_\eta T(x^*)$ exists in the direction η, with $\partial_\eta T(x^*) = \max\langle p^\pm, \eta\rangle$ $(\min\langle p^\pm, \eta\rangle)$ if the vector $p^+ - p^-$ is directed from G^- to G^+ (from G^+ to G^-).

3. DISPERSAL AND EQUIVOCAL SINGULAR POINTS

Various types of singular surfaces, at whose points optimal motions have different properties, are known for value functions $T(x)$ in the theory of differential games [3, 7]. The classification of these surfaces is based on an analysis the behaviour of optimal paths in the neighbourhood of singular points, also allowing for the possibility of singular optimal motions on the singular surface itself.

We shall extend the concepts of dispersal and equivocal singular points [3, 7] to the case of a function $T(x)$. To that end, we will first define the corresponding types of simple singular points and clarify the geometrical meaning of the definitions.

Definition 4. A point $x^* \in \Sigma(T)$ is called a *dispersal point* if

$$p^+ \neq p^-, \quad \langle f^+, p^+ - p^-\rangle > 0, \quad \langle f^-, p^+ - p^-\rangle < 0$$

A simple singular point x^* is a dispersal point if characteristics from the adjacent regular domains G^+ and G^- leave it at a non-zero angle to the separating hypersurface Γ (Fig. 1a).

Definition 5. A point $x^* \in \Sigma(T)$ is called an *equivocal point* (relative to the second player) if

$$p^+ \neq p^-, \quad \langle f^+, p^+ - p^-\rangle > 0, \quad \langle f^-, p^+ - p^-\rangle \geq 0$$

and, moreover, a control vector $\hat{u}(x^*) \in P$ exists such that

$$\langle p^+, \hat{f} \rangle = \langle p^-, \hat{f} \rangle = -1$$

where $\hat{f} := f(x^*, \hat{u}(x^*), \upsilon^-)$ $(f(x^*, \hat{u}(x^*), \upsilon^+))$, when the vector $p^+ - p^-$ is directed from G^- to G^+ (from G^+ to G^-).

A characteristic from the regular domain G^- arrives at an equivocal point, but a characteristic from the regular domain G^+ leaves it (at a non-zero angle to Γ); in addition, a control for the first player exists that guarantees motion at a velocity equal to -1 on the surface Γ in the case when the second player uses a limiting control corresponding to the domain G^- (Fig. 1b).

A smooth hypersurface is said to be a dispersal (equivocal) hypersurface if it consists entirely of dispersal (equivocal) points.

4. THE SUFFICIENT CONDITIONS FOR STABILITY

We will now formulate the main result of this paper.

Theorem. Suppose $f(x, u, \upsilon) = \varphi(x, u) + \psi(x, \upsilon)$ and let the set $\psi(x, Q) = \{\psi(x, \upsilon): \upsilon \in Q\}$ be a line segment in R^n. If $x^* \in \Sigma(T)$ is a dispersal point or an equivocal point, and moreover $\psi(x^*, \upsilon^+) \neq (x^*, \upsilon^-)$, then the stability conditions (1.4) are satisfied there.

Proof. Consider a point $x^* \in \Sigma(T)$ and assume that the vector $p^+ - p^-$ is directed from G^- to G^+.

Taking into account the expression for the directional derivative at a simple singular point and the separated dynamics of system (1.1), we rewrite the stability conditions (1.4) at the point x^* as follows:

1) for vector $\upsilon \in Q$ a vector $\varphi_\upsilon \in \operatorname{co}\varphi(x^*, P)$ exists such that $\langle p^\pm, \varphi_\upsilon + \psi(x^*, v)\rangle \leq -1$ (u-stability);

2) for any vector $u \in P$ a vector $\psi_u \in \psi(x^*, Q)$ exists such that

$$\max\{\langle p^+, \varphi(x^*, u) + \psi_u\rangle, \langle p^-, \varphi(x^*, u) + \psi_u\rangle\} \geq -1$$

(υ-stability).

Suppose

$$\mathbf{Q}(x, p) := \underset{\upsilon \in Q}{\operatorname{argmax}}\langle p, \psi(x, \upsilon)\rangle$$

Since $\psi(x, Q)$ is a line segment in R^n, it follows that the set $\psi(x, \mathbf{Q}(x, p))$ is either a singleton and contains one of the endpoints of $\psi(x, Q)$, or it is the entire segment $\psi(x, Q)$. By definition, we have $V_T^\mp(x) \in \mathbf{Q}(x, DT(x))$. Since $G^\pm$ are regular domains, it follows that $\psi(x, V_T^\mp(x)) \in C^1(G^\pm)$. Consequently, $\psi(x, V_T^\mp(x))$ is an endpoint of the segment $\psi(x, Q)$ for any $x \in G^\pm$.

We introduce the following notation.

$$\varphi^\pm := \varphi(x^*, u^\pm), \quad \psi^\pm := \psi(x^*, \upsilon^\pm)$$

It follows from the definition of the vector $\upsilon^\pm$ and from the condition $\psi^+ \neq \psi^-$ that $\psi^\pm$ are different endpoints of $\psi(x^*, Q)$. Thus, we have the representation

$$\psi(x^*, Q) = \{\lambda\psi^+ + (1-\lambda)\psi^- : \lambda \in [0, 1]\}$$

We will first prove that condition 1 holds at the point x^*.

For any $\upsilon \in Q$, let us find a number $\lambda_\upsilon \in [0, 1]$ such that

$$\psi(x^*, \upsilon) = \lambda_\upsilon\psi^+ + (1-\lambda_\upsilon)\psi^-$$

Suppose x^* is a dispersal point. Set

$$\varphi_\upsilon := \lambda_\upsilon\varphi^+ + (1-\lambda_\upsilon)\varphi^-$$

Since $\varphi^\pm \in \varphi(x^*, P)$, it follows that $\varphi_\upsilon \in \operatorname{co}\varphi(x^*, P)$. We have

$$\langle p^\pm, \varphi_\upsilon + \psi(x^*, \upsilon)\rangle = \lambda_\upsilon\langle p^\pm, \varphi^+ + \psi^+\rangle + (1-\lambda_\upsilon)\langle p^\pm, \varphi^- + \psi^-\rangle \tag{4.1}$$

By the definition of a dispersal point

$$\langle f^+, p^+ - p^- \rangle > 0, \quad \langle f^-, p^+ - p^- \rangle < 0$$

Consequently

$$\langle p^-, f^+ \rangle < \langle p^+, f^+ \rangle = -1, \quad \langle p^+, f^- \rangle < \langle p^-, f^- \rangle = -1 \tag{4.2}$$

Since $f^{\pm} = \varphi^{\pm} + \psi^{\pm}$, it follows from (4.1) and (4.2) that

$$\langle p^{\pm}, \varphi_v + \psi(x^*, v) \rangle \le -1 \tag{4.3}$$

this proves u-stability for a dispersal point.

Now suppose x^* is an equivocal point. Put

$$\varphi_v := \lambda_v \varphi^+ + (1 - \lambda_v)\hat{\varphi}, \quad \hat{\varphi} := \varphi(x^*, \hat{u}(x^*))$$

where $\hat{u}(x^*)$ is a singular control of the first player as in the definition of an equivocal point. Since $\varphi^+, \hat{\varphi} \in (x^*, P)$, it follows that $\varphi_v \in \mathrm{co}\,\varphi(x^*, P)$. In addition, an equality analogous to (4.1) is true with φ^- replaced by $\hat{\varphi}$.

By the definition of an equivocal point,

$$\langle f^+, p^+ - p^- \rangle > 0, \quad \langle p^-, \hat{f} \rangle = \langle p^+, \hat{f} \rangle = -1$$

We have

$$\langle p^-, f^+ \rangle < \langle p^+, f^+ \rangle = -1, \quad f^+ = \varphi^+ + \psi^+, \quad \hat{f} = \hat{\varphi} + \psi^-$$

Consequently, inequality (4.3) is true. This proves u-stability for an equivocal point.

The truth of condition 2 at the point x^* follows from the inequality

$$\langle p^{\pm}, \varphi(x^*, u) + \psi^{\pm} \rangle \ge \langle p^{\pm}, \varphi^{\pm} + \psi^{\pm} \rangle = -1$$

if we put

$$\psi_u := \psi^{\pm}$$

Note that the assumption that the vector $p^+ - p^-$ is directed from G^- to G^+ guarantees satisfaction of the v-stability condition for dispersal and equivocal points without the restrictions on the dynamics of system (1.1) described in the conditions of the theorem. Similarly, if the vector $p^+ - p^-$ is directed from G^+ to G^-, the conditions of the theorem are not used to prove u-stability at dispersal or equivocal points.

In some cases, the theorem enables one to justify the application of Isaacs' technique [3] to look for the value function of a game using fields of characteristics and the singular surfaces just constructed.

5. THE BRACHISTOCHRONE GAME PROBLEM

As an example, let us apply the theorem to the brachistochrone game problem, in which singular curves of the types described appear.

The branchistochrone game problem was considered by Isaacs [3], and his solution was improved and expanded in [8]. The game problem investigated in the present paper differs in its formulation from Isaacs' problem in the form of the terminal set and the vectogram $\psi(x, Q)$ of the second player.

Formulation of the problem. Consider the minimum-time differential game

$$\begin{gathered} \dot{x}_1 = \sqrt{x_2}\cos u, \quad \dot{x}_2 = \sqrt{x_2}\sin u + wv \\ u \in P = [0, 2\pi], \quad v \in Q = [-1, 1], \quad t \ge 0, \quad x_0 \in R_+^2 \end{gathered} \tag{5.1}$$

where R_+^2 is the upper half-plane.

The first (second) player minimizes (maximizes) the time needed to reach the terminal set $M = [-d, 0] \times [0, h]$, where $d, h > 0$. The second player's success depends on the number $w > 0$. If $w = 0$ we obtain the dynamics of the classical brachistochrone problem [9].

First, using Isaacs' technique, we define a function $T: R_+^2 \to [0, \infty]$. Then, applying the theorem on the sufficient conditions for stability, we shall show that $T(x)$ is a value function in the branchistochrone game problem.

Since the right-hand side of system (5.1) is independent of x_1, the game is symmetrical about the vertical line $x_1 = -d/2$. From now on all constructions will be carried out for the half-plane $x_1 \geqslant -d/2$. The results of the computations will be illustrated for different values of h at $w = 2$.

The Bellman–Isaacs equation. Any function in a regular domain is uniquely defined as the solution of a certain boundary-value problem for the Bellman–Isaacs equation (1.5). This solution may be found by the method of classical characteristics. Therefore, the first step in constructing $T(x)$ will be integration of the characteristic system (2.1).

Let us write Eq. (5.1) for the branchistochrone game problem. We have

$$H(x, p) = \min_{u \in [0, 2\pi]} \max_{\upsilon \in [-1, 1]} [p_1\sqrt{x_2}\cos u + p_2(\sqrt{x_2}\sin u + w\upsilon)] \tag{5.2}$$

Extremal controls u^0 and υ^0 achieving the minimum and maximum in (5.2) are defined by the formulae

$$\cos u^0 = -p_1/\|p\|, \quad \sin u^0 = -p_2/\|p\|, \quad \|p\| = \sqrt{p_1^2 + p_2^2}$$

$$\upsilon^0 \in \mathbf{Q}(x, p) = \begin{cases} \operatorname{sign} p_2, & \text{if} \quad p_2 \neq 0 \\ [-1, 1], & \text{if} \quad p_2 = 0 \end{cases}$$

Consequently, $H(x, p) = -\sqrt{x_2}\|p\| + w|p_2|$, and Eq. (1.5) for the function $T(x)$ becomes

$$-\sqrt{x_2}\|DT\| + w|\partial T/\partial x_2| = -1 \tag{5.3}$$

where $DT = (\partial T/\partial x_1, \partial T/\partial x_2)$ is the vector of partial derivatives of $T(x)$.

The characteristic system. On the assumption that $p_2 \neq 0$, $x_2 > 0$, the characteristic system for Eq. (5.3) in reverse time may be written as

$$x_1' = \sqrt{x_2}\frac{p_1}{\|p\|}, \quad p_1' = 0, \quad x_2' = \sqrt{x_2}\frac{p_2}{\|p\|} - w\mu_2, \quad p_2' = -\frac{\|p\|}{2\sqrt{x_2}} \tag{5.4}$$

where

$$z' = dz/d\tau, \quad \tau = \text{const} - t, \quad \mu_2 = \operatorname{sign} p_2$$

Suppose the initial conditions for the characteristic system are given as

$$x(0, s) = \xi(s), \quad p(0, s) = \zeta(s), \quad s \in S \tag{5.5}$$

where the function $\xi(s)$ defines a smooth curve parametrically

$$\Gamma = \{x = \xi(s) : s \in S\}$$

The phase curve $x(\tau, s)$ (for fixed s) is a characteristic. Varying the parameter s, we obtain a family of characteristics emanating from the points of the curve Γ.

Let us integrate system (5.4). Since $p_1' = 0$, we will henceforth let the symbol p_1 denote a constant, determined from the initial data of system (5.4).

Two cases will be considered. First let $p_1 = 0$. The first and third equations of system (5.4) become

$$x_1' = 0, \quad x_2' = (\sqrt{x_2} - w)\mu_2.$$

The yield first integrals

$$x_1 = C_1, \quad \mu_2\tau - 2(\sqrt{x_2} + w\ln(\sqrt{x_2} - w)) = C_2 \tag{5.6}$$

Equation (5.3) is one more first integral

$$(w - \sqrt{x_2})\mu_2 p_2 = -1$$

In that case the characteristics will be vertical straight lines. By virtue of the initial conditions (5.5), Eqs (5.6) may be rewritten as

$$x_1 = \xi_1(s), \quad \mu_2\tau = 2(\sqrt{x_2} + w\ln(\sqrt{x_2} - w)) + C_2(s)$$

The function $C_2(s)$ is determined by substituting the initial data (5.5) into the left-hand side of the second equation in (5.6). If the function $\xi_1(s)$ is invertible, the second equation will define the function $\tau = T(x)$ in the domain covered by the vertical characteristics.

Now let $p_1 \neq 0$. Equation (5.3) is a first integral

$$-\sqrt{x_2}\|p\| + w|p_2| + 1 = 0$$

of the characteristic system, defining the relation between the quantities x_2 and p_2. Using this relation, we group Eqs (5.4) as a system

$$x_1' = \frac{x_2 p_1(w^2 - x_2)}{-x_2 + \sigma w R(x_2, p_1)}, \quad x_2' = \frac{\mu_2 \sigma R(x_2, p_1)(x_2 - w^2)}{-x_2 + \sigma w R(x_2, p_1)} \tag{5.7}$$

where

$$\sigma := \operatorname{sign}(wp_1^2 - |p_2|), \quad R(x_2, p_1) := \sqrt{x_2(1 + p_1^2(w^2 - x_2))}$$

and an equation

$$p_2' = -\|p\|^2/(2w|p_2| + 2) \tag{5.8}$$

In the domains where σ is a constant, we find a first integral of system (5.7):

$$x_1 + \mu_1\mu_2\sigma[\lambda(p_1)\arcsin\sqrt{x_2/\lambda(p_1)} - \sqrt{x_2(\lambda(p_1) - x_2)}] = D_1$$

and thus also of system (5.4), where

$$\mu_i := \operatorname{sign} p_i, \quad i = 1, 2, \quad \lambda(p_1) := w^2 + 1/p_1^2$$

and D_1 is a constant.

Integration of Eq. (5.8) determines yet another first integral of system (5.4):

$$\tau + w\mu_2 \ln\|p\|^2 + p_1^{-1}\operatorname{arctg}(p_2/p_1) = D_2$$

The initial data (5.5) determine the values of the constants D_1 and D_2 as functions of $s \in S$, so that we can write the system of equations

$$\begin{aligned} &-\sqrt{x_2}\|p\| + w\mu_2 p_2 + 1 = 0 \\ &x_1 + \mu_1\mu_2\sigma[\lambda(p_1)\arcsin\sqrt{x_2/\lambda(p_1)} - \sqrt{x_2(\lambda(p_1) - x_2)}] = D_1(s) \\ &\tau + w\mu_2\ln\|p\|^2 + p_1^{-1}\operatorname{arctg}(p_2/p_1) = D_2(s) \\ &p_1 = \zeta_1(s) \end{aligned} \tag{5.9}$$

which is an algebraic system of four equations in the four unknowns p_1, p_2, s and τ. It implicitly defines the function $\tau = T(x)$ in the domain covered by the characteristics.

The characteristic system (5.4) has been completely integrated.

The complications in the subsequent analytical investigation of the problem are due to the impossibility of explicitly defining the function $T(x)$.

The construction of primary families of characteristics for $h > w^2$. We will first consider the case $h > w^2$. Let us find the admissible zone $\Gamma_0 \subset \partial M$ [3] (taking the aforementioned symmetry of the problem into consideration). We have

$$\Gamma_0 = \{x_1 = 0, 0 < x_2 \leq h\} \cup \{-d/2 \leq x_1 \leq 0, x_2 = h\}$$

Put $T(x) = 0$ on Γ_0. We shall construct primary families of characteristics emanating from the horizontal and vertical parts of the set Γ_0, as well as from the point $(0, h)$.

The characteristics emanating from the horizontal part of Γ_0 at $\mu_2 = +1$ are vertical straight lines; they define the function

$$T(x) := 2\sqrt{x_2} - \sqrt{h} + w\ln((\sqrt{x_2} - w)/(\sqrt{h} - w)) \tag{5.10}$$

at points of the vertical strip $\{(x_1, x_2): -d/2 \leqslant x_1 \leqslant 0, x_2 \geqslant h\}$.

We now construct two smooth families of characteristics. The first emanates from the right vertical part of the set Γ_0 with $\mu_2 = -1$, and the second, from the point $(0, h)$ with $\mu_2 = +1$. These particular values of μ_2 are chosen for heuristic reasons.

The resulting families partly overlap; they define functions $T_1(x)$ and $T_2(x)$ in certain domains Ω_1 and Ω_2 respectively.

The first family of characteristics is bounded below by a smooth curve $\mathcal{B}$ consisting of semi-permeable curve [3] defined by the equation

$$x_1 = B_*(x_2) := w^2 \arcsin\frac{\sqrt{x_2}}{w} - \sqrt{x_2(w^2 - x_2)}, \quad x_2 \in [0, w^2]$$

and the ray $\{(x_1, x_2): x_1 \geqslant \pi w^2/2, x_2 = w^2\}$. The curve $\mathcal{B}$ will be a barrier [3].

The second family is bounded below by the curve $\tilde{\mathcal{S}}$ defined by the condition $p_2 = +0$ for motion along the characteristics.

Figure 2 illustrates numerical constructions of the primary families of characteristics emanating in reverse time from the admissible zone Γ_0 for parameter values $h = 9$, $w = 2$. The first and second families of characteristics are labelled *1* and *2* respectively.

Construction of singular curve for $h > w^2$. On the basis of the primary families of characteristics, we shall now construct a singular curve separating the domain above the barrier $\mathcal{B}$ into two sets. Above

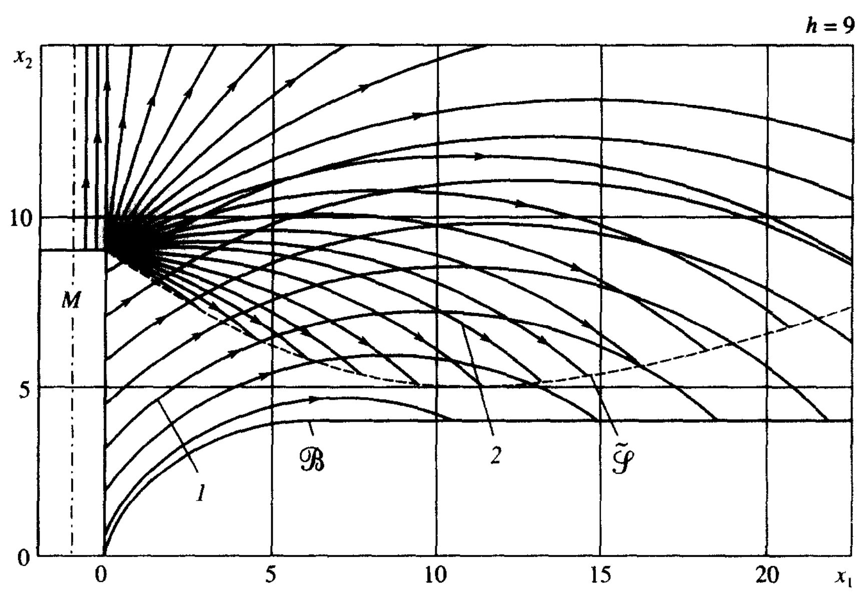


Fig. 2

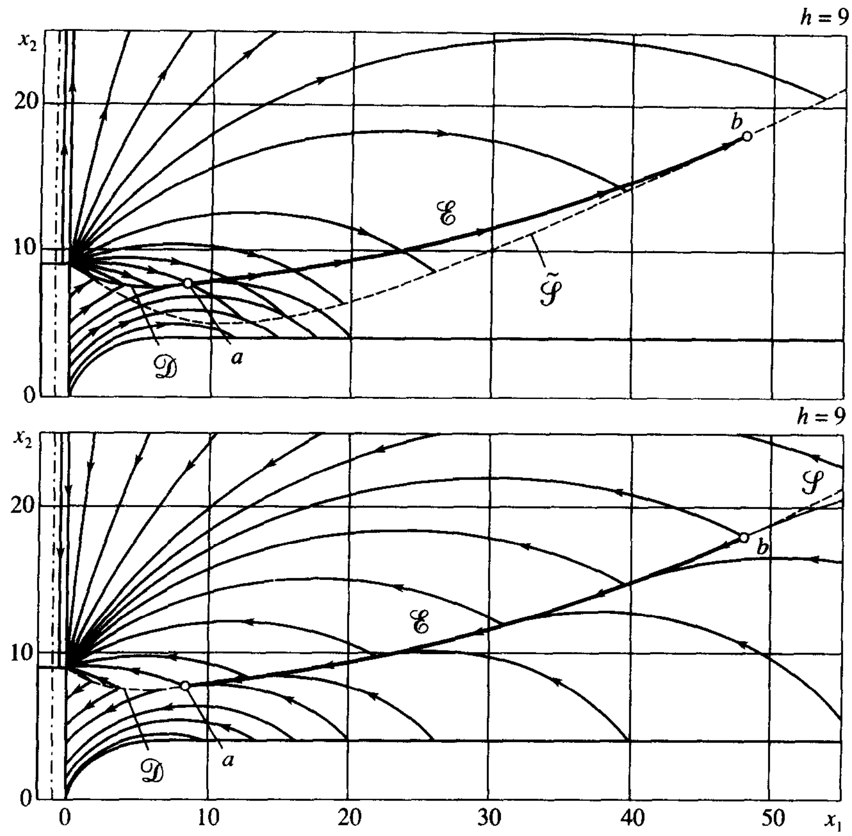


Fig. 3

the singular curve, the function $T(x)$ being constructed will coincide with the function $T_2(x)$; below, it will partly coincide with $T_1(x)$, and will be partly defined by a secondary family of characteristics emanating from the points of the singular curve.

Let

$$\tilde{\mathcal{D}} = \{x \in \Omega_1 \cap \Omega_2 \colon T_1(x) = T_2(x)\}$$

If $h > w^2 + \Delta_w$ and Δ_w is some positive number, the curve $\tilde{\mathcal{D}}$ is tangent to one of the characteristics of the (critical) first family. For values of h near w^2 the curve $\tilde{\mathcal{D}}$ is tangent to the barrier $\mathcal{B}$. In both cases, denote the point of contact by $a = (a_1, a_2)$, and define

$$\mathcal{D} = \{(x_1, x_2) \in \tilde{\mathcal{D}} \colon x_1 \le a_1\}$$

The parts of the characteristics of both families after their intersection with the curve $\mathcal{D}$ will be omitted.

If $a \notin \mathcal{B}$, we continue $\mathcal{D}$ by a curve $\mathcal{E}$, motion along which (in reverse time) is given by the equations

$$x_1' = -\sqrt{x_2}\cos\hat{u}(x_1, x_2), \quad x_2' = -\sqrt{x_2}\sin\hat{u}(x_1, x_2) + w$$

$$\hat{u}(x_1, x_2) = \operatorname{arctg}\frac{p_2}{p_1} + \arccos\frac{wp_2 - 1}{\sqrt{x_2}\|p\|}, \quad p_i(x) = \frac{\partial T_2}{\partial x_i}(x), \quad i = 1, 2$$

The control $\hat{u}(x)$ is found rom the equality

$$p_1(x)\sqrt{x_2}\cos\hat{u}(x) + p_2(x)(\sqrt{x_2}\sin\hat{u}(x) - w) = -1$$

If $a \in \mathcal{B}$, the curve $\mathcal{E}$ will emanate from some point $a^* = (a_1^*, a_2^*)$ defined by the condition that the vector

$$(\sqrt{a_1^*}\cos\hat{u}(a^*), \sqrt{a_2^*}\sin\hat{u}(a^*) - w)$$

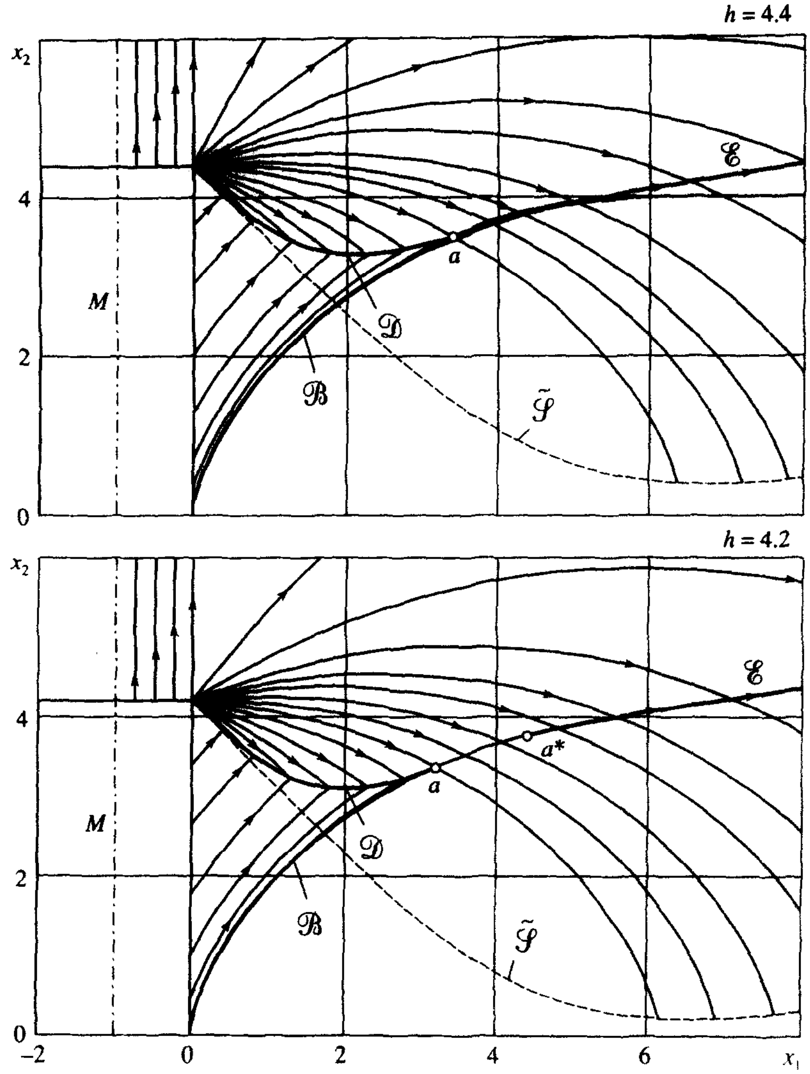


Fig. 4

and the curve $\mathcal{B}$ are tangent at the point a^*.

The curve $\mathcal{E}$ is continuable up to a point $b = (b_1, b_2)$ where it is tangent to the curve $\tilde{\mathcal{S}}$. Let

$$\mathcal{S} = \{(x_1, x_2) \in \tilde{\mathcal{S}}: x_1 \geq b_1\}$$

The parts of the characteristics of the second family after their intersection with the curve $\mathcal{E}$ will be omitted.

The upper part of Fig. 3 illustrates the results of numerical constructions (in reverse time) of the curves $\mathcal{D}$ and $\mathcal{E}$ for parameter values $h = 9$, $w = 2$. The parts of the characteristics after intersection with $\mathcal{D}$ are not shown.

We will now specify initial data for the system of characteristics (5.4) on the curves $\mathcal{E}$, $\mathcal{S}$ at $\mu_2 = -1$, based on the continuity conditions and on the properties of equivocality on $\mathcal{E}$ and differentiability on $\mathcal{S}$. We construct the secondary family of characteristics, which is not continuable below the line $x_2 = w^2$ and completely covers the domain between the barrier $\mathcal{B}$ and the curve $\mathcal{E}\mathcal{S}$.

In the lower part of Fig. 3 we show numerical constructions of all the families of characteristics in forward time at $h = 9$, $w = 2$, and the curve $\mathcal{D}\mathcal{E}\mathcal{S}$ (the case $h > w^2 + \Delta_w$). The parts $\mathcal{D}$ and $\mathcal{S}$ are shown as dashed curves.

Figure 4 shows, in greater detail, the case of closely situated curves $\mathcal{D}$, $\mathcal{E}$ and of the curve $\mathcal{B}$ for parameter values $h = 4.4$, $w = 2$ (the upper part of Fig. 4), as well as the case of intersection (along the tangent) of the curve $\mathcal{D}$ with the barrier $\mathcal{B}$ and the construction of the curve $\mathcal{E}$ from a point $a^* \in \mathcal{B}$ for $h = 4.2$, $w = 2$ (the lower part of Fig. 4).

Value function of the game when $h > w^2$. The right-hand side of system (5.1) does not satisfy a Lipschitz condition as a function of x. Nonetheless, within the framework of the problem currently under consideration, the definition of the value function using pencils of constructive motions and Eq. (1.3) remains unchanged.

We define the function $T(x)$ as follows. At points of the vertical strip $\{(x_1, x_2): -d/2 \leqslant x_1 \leqslant 0, x_2 \geqslant h\}$, $T(x)$ is given by formula (5.10); $T(x) = T_2(x)$ above the curves $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{S}$; below the curve $\mathcal{D}$ and below the critical characteristic (if it exists), define $T(x) = T_1(x)$; in the remaining domain, $T(x)$ is defined by he secondary family of characteristics. On the barrier $\mathcal{B}$, we define $T(x)$ by continuity. In the domain below the barrier curve, we set $T(x) = \infty$.

By construction, $\mathcal{D}$ is a dispersal curve for $T(x)$, and $\mathcal{E}$ is an equivocal curve. The curve $\mathcal{S}$ is called a switching curve. It consists of simple singular points for which the equality $p^+(x) = p^-(x)$ holds.

On the set $x_1 < -d/2, x_2 \geqslant 0$, the function $T(x)$ is defined by symmetry with respect to the straight line $x_1 = -d/2$.

The smooth branches of $T(x)$ are solutions of Eq. (5.3); by the theorem proved previously, the dispersal and equivocal nature of the splicing guarantee that the stability conditions (1.4) are satisfied at points where the function is not smooth.

We now introduce the notation

$$\Omega := \{x \in R_+^2: T(x) < \infty\}$$

Using the stability properties of the function $T(x)$ at interior points of the set Ω, we will show that for $h > w^2$ the function $T(x)$ just constructed is the value function in the brachistochrone game problem.

In the class of positional strategies, it is always true [1] that

$$\inf_U \sup J(X_1(x, U)) \geq \sup_V \inf J(X_2(x, V)), \quad x \in R_+^2$$

In the set $R_+^2 \backslash \Omega$, the second player has a deflecting strategy $V(x) = -1$. Thus,

$$\sup_V \inf J(X_2(x, V)) = \infty$$

and therefore Eq. (1.3) holds for the points $x \in R_+^2 \backslash \Omega$.

On the set $\Omega \backslash M$ it will suffice to prove that

$$\inf_U \sup J(X_1(x, U)) \leq T(x) \leq \sup_V \inf J(X_2(x, V)) \tag{5.11}$$

1. For arbitrary $c > 0$, we find the point $x^c = (x_1^c, x_2^c) \in \mathcal{B}$ defined by the condition $T(x^c) = c$. Suppose

$$W_c := \{x \in R_+^2: T(x) \leq c\}, \quad \tilde{W}_c := \{x \in W_c: x_2 \geq x_2^c\}$$

On the set R_+^2 we define a function

$$T_c(x) = \begin{cases} T(x) - c, & \text{if} \quad x \in \Omega \backslash W_c \\ 0, & \text{if} \quad x \in \tilde{W}_c \\ \infty, & \text{if} \quad x \in (R_+^2 \backslash \Omega) \cup (W_c \backslash \tilde{W}_c) \end{cases}$$

The boundary of the set W_c is labelled *1* in Fig. 5 and the set $\tilde{W}_c$ is hatched.

Suppose

$$\tilde{\Omega} := \{x \in R_+^2: T_c(x) < \infty\}$$

From any point of the set $\tilde{\Omega}$, the first player guarantees arrival at the set $\tilde{W}_c$ in a finite time. The function $T_c(x)$ is u- and v-stable at any interior point of the set $\tilde{\Omega} \backslash \tilde{W}_c$, since that is the case for $T(x)$. In addition, the right-hand side of system (5.1) satisfies a Lipschitz condition with respect to x above the straight line $x_2 = x_2^c/2$. Using the facts just listed, one establishes that $T_c(x)$ is a value function in the minimum-time game problem with terminal set $\tilde{W}_c$ and game space situated above the straight line $x_2 = x_2^c/2$.

2. We will now consider the problem of approaching the set M from a point $x_* \in \Omega \backslash M$.

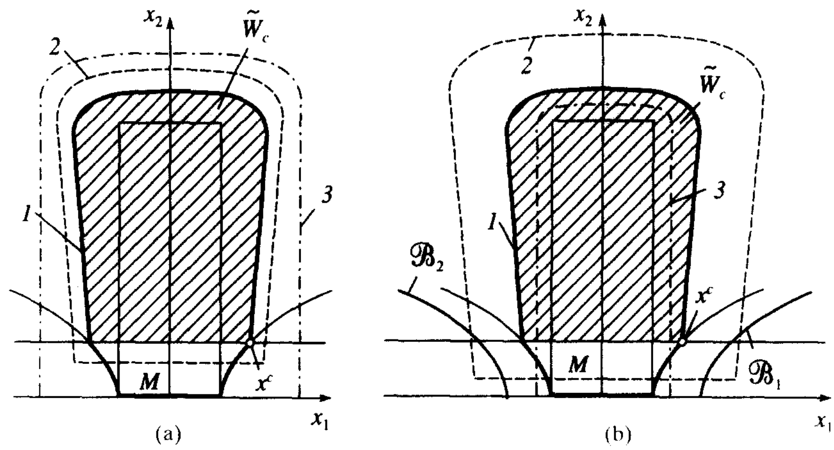


Fig. 5

For arbitrary $\varepsilon > 0$ and $\delta_1 > 0$, we define $\delta = \delta_1/2$ and find a number $c > 0$ such that the open δ-neighbourhood $\tilde{W}_c^\delta$ of the set $\tilde{W}_c$ is contained in the open δ_1-neighbourhood M_{δ_1} of the set M. The boundary of the set $\tilde{W}_c^\delta$ is labelled *2* in Fig. 5(a) that of the set M_{δ_1} is labelled *3*.

Since $T_c(x)$ is a value function for the minimum-time problem with terminal set $\tilde{W}_c$, a strategy $\tilde{U}_\varepsilon$ of the first player exists [6] that guarantees approach to the set $\tilde{W}_c^\delta$ in a time $T_c(x_*) + \varepsilon$ in a discrete control scheme with stepsize $\Delta \leqslant \tilde{\Delta}(\delta)$, where $\tilde{\Delta}(\delta)$ is a positive number such that $\tilde{\Delta}(\delta) \to 0$ as $\delta \to 0$.

Since $\tilde{W}_c^\delta \subset M_\delta$ and $T_c(x^*) < T(x_*)$, it follows that the strategy $\tilde{U}_\varepsilon$ also guarantees approach to the set M_{δ_1} in a time $T(x_*) + \varepsilon$ in a discrete control scheme with stepsize $\Delta \leqslant \tilde{\Delta}(\delta)$.

Let $\delta_1 \to 0$. Using the compactness of the bundle $X_1(x_*, \tilde{U}_\varepsilon)$ of constructive motions [1], we obtain

$$\sup J(X_1(x_*, \tilde{U}_\varepsilon)) \leq T(x_*) + \varepsilon$$

This implies the left inequality in (5.11).

3. We will now consider the problem of evading the set M when approaching from a point $x^* \in \Omega \backslash M$. For arbitrary $\varepsilon > 0$, define $c = \varepsilon/2$. We introduce the notation

$$t_* := T_c(x_*) - \varepsilon/2$$

Since $T_c(x)$ is a value function for the minimum-time problem with terminal set $\tilde{W}_c$, a strategy $\tilde{V}_\varepsilon$ and numbers $\delta > 0, \tilde{\Delta} > 0$ exist [6] such that the strategy $\tilde{V}_\varepsilon$ guarantees evasion of the closed δ-neighbourhood $\tilde{W}_c^\delta$ of $\tilde{W}_c$ up to time t_* in a discrete control scheme with stepsize $\Delta \leqslant \tilde{\Delta}$. The boundary of the set $\tilde{W}_c^\delta$ is labelled *2* in Fig. 5(b).

Note that when setting up the extremal equations [6] for the second player that define the strategy $\tilde{V}_\varepsilon$, one can confine ones attention to the numbers ± 1.

Let $\mathcal{B}_1$ be the curve obtained by displacing the barrier $\mathcal{B}$ to the right parallel to the horizontal axis, in such a way that the point of intersection of the straight line $x_2 = x_2^c$ and the curve $\mathcal{B}_1$ lies in the set $\tilde{W}_c^\delta$. The symbol $\mathcal{B}_2$ will denote the curve symmetric to $\mathcal{B}_1$ with respect to the vertical line $x_1 = -d/2$. The curves $\mathcal{B}_1$ and $\mathcal{B}_2$ are shown in Fig. 5(b). Using the control $v = -1$, the second player refrains from motion in the domain below curve $\mathcal{B}_1(\mathcal{B}_2)$.

On the basis of the strategy $\tilde{V}_\varepsilon$, we define a strategy V_ε as follows. Define $V_\varepsilon(x) = -1$ at points strictly below the curve $\mathcal{B}_1$ and $V_\varepsilon(x) = \tilde{V}_\varepsilon(x)$ at other points of the half-plane $x_1 \geqslant -d/2$. The strategy V_ε is defined symmetrically in the half-plane $x_1 \leqslant -d/2$.

Choose a number $\delta_1 > 0$ in such a way that the δ_1-neighbourhood M_{δ_1} of the set M does not intersect the curve $\mathcal{B}_1$. The boundary of the set M_{δ_1} is labelled 3 in Fig. 5(b). We will show that the strategy V_ε guarantees evasion of the set M_{δ_1} up to a time t_* in a discrete control scheme with stepsize $\Delta \leqslant \tilde{\Delta}$.

Define a partition of the semiaxis $t \geqslant 0$ by intervals $[t_i, t_{i+1})$ and find a number N such that $t_* \notin [t_N, t_{N+1})$. Consider an arbitrary motion $x(t)$ generated by the strategy V_ε in a discrete control scheme.

Suppose there is an instant of time t_j, $0 \leqslant j \leqslant N$, such that the point $x(t_j)$ lies strictly beneath the curve $\mathcal{B}_1(\mathcal{B}_2)$ and $x(t) \notin M_{\delta_1}$ for $t \leqslant t_j$. Then, by the definition of the strategy V_ε, the point $x(t)$ will remain below the curve $\mathcal{B}_1(\mathcal{B}_2)$ for any $t > t_j$ and will not reach M_{δ_1} for an infinite interval of time.

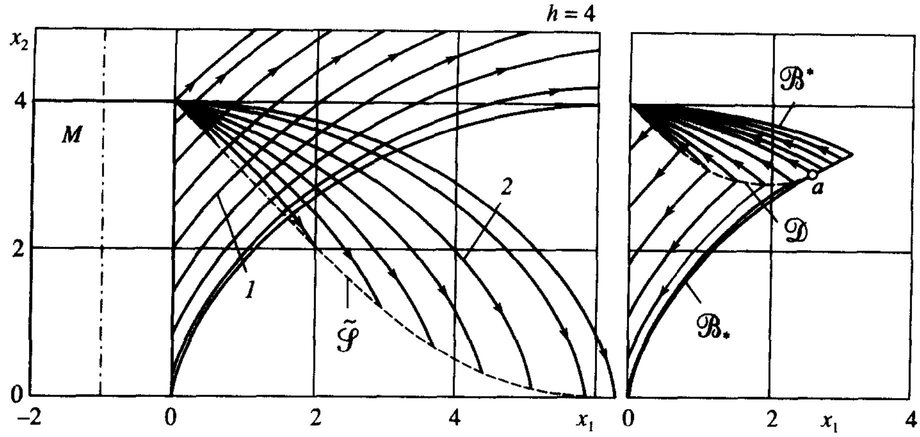


Fig. 6

Now suppose that for all times t_i, $0 \leqslant i \leqslant N$, the point $x(t_i)$ does not lie beneath the curves $\mathcal{B}_1$ and $\mathcal{B}_2$. Consequently, at each time t_i the second player's control is chosen in accordance with the strategy $\tilde{V}_\varepsilon$. In that case the path $x(t)$ cannot intersect the straight line $x_2 = x_2^c$ up to time t_*.

Suppose the contrary. Let $\bar{t} \leqslant t_*$ be the first time at which the point $x(\bar{t})$ belongs to the straight line $x_2 = x_2^c$. Let j be a number such that $\bar{t} \in (t_j, t_{j+1})$, $0 \leqslant j \leqslant N$. Since the strategy $\tilde{V}_\varepsilon$ evades the closed set $\tilde{W}_c^\delta$ up to time t_*, it follows that $x(\bar{t})$ will lie either strictly below $\mathcal{B}_1$ or strictly below $\mathcal{B}_2$. In the interval $[t_j, t_{j+1})$ we have $V_\varepsilon(x(t)) = 1$, since otherwise, i.e., if $V_\varepsilon(x(t)) = -1$, the point $x(t_{j+1})$ would lie below $\mathcal{B}_1$ or below $\mathcal{B}_2$.

Since $\sqrt{x_2(\bar{t})} < w$, it follows that

$$\dot{x}_2(t) = \sqrt{x_2(\bar{t})}\sin u + w > 0$$

Therefore, in the interval $[t_j, \bar{t})$ the motion takes place below the straight line $x_2 = x_2^c$. Since the point $x(t_j)$ is not below $\mathcal{B}_1$ or $\mathcal{B}_2$, there are a time $\tilde{t} \leqslant t_j$ such that $x(\tilde{t}) \in \tilde{W}_c^\delta$. But this contradicts the definition of the strategy $\tilde{V}_\varepsilon$. Consequently, the motion $x(t)$ cannot intersect the straight line $x_2 = x_2^c$ up to a time t_*.

Thus, the motion $x(t)$ takes place above the line $x_2 = x_2^c$ up to a time t_*. Since at each instant of time t_i, $0 \leqslant i \leqslant N$, the second player's control is chosen in accordance with the strategy V_ε, it follows that $x(t) \notin \tilde{W}_c^\delta$ for $t \leqslant t_*$. Consequently, the motion $x(t)$ will not reach the set M_{δ_1} up to time t_*.

Since $c = \varepsilon/2$, it follows that $t_* = T(x_*) - \varepsilon$. We thus deduce that the strategy V_ε evades the set M_{δ_1} up to time $T(x_*) - \varepsilon$. Hence the bundle $X_2(x_*, V_\varepsilon)$ of constructive motions satisfies the inequality

$$\inf J(X_2(x_*, V_\varepsilon)) \geq T(x_*) - \varepsilon$$

This implies the right-hand inequality in (5.11).

The value function of the game when $h \leqslant w^2$. In the $h \leqslant w^2$, we define $T(x) = T_2(x)$ above the curve $\mathcal{D}$ and $T(x) = T_1(x)$ below the curve $\mathcal{D}$. The functions $T_1(x)$, $T_2(x)$ and the curve $\mathcal{D}$ are constructed as in the case $h > w^2$.

Fields of characteristics for parameter values $h = 4$, $w = 2$ are shown in the left-hand part of Fig. 6. The entire singular curve $\mathcal{D}$ is a dispersal curve (the right-hand part of Fig. 6).

The barrier curve consists of a part $\mathcal{B}_*$ of the semi-permeable curve $x_1 = B_*(x_2)$ lying in the strip $0 \leqslant x_2 \leqslant x_2^h$ and a part $\mathcal{B}_*$ of the semi-permeable curve $x_1 = B^*(x_2, h)$: $= -B_*(x_2) + B_*(h)$ lying in the strip $x_2^h \leqslant x_2 \leqslant w^2$, where the number x_2^h is defined by the equation $B_*(x_2) = B^*(x_2, h)$. On the curve $\mathcal{B}_*$ and $\mathcal{B}^*$, $T(x)$ is defined by continuity. At all other points we set $T(x) = \infty$.

On the set $x_1 < -d/2, x_2 \geqslant 0$, $T(x)$ is defined symmetrically with respect to the straight line $x_1 = -d/2$.

The dispersal nature of the splicing of the smooth branches of $T(x)$ guarantees satisfaction of the stability conditions (1.4) at points where the function is not smooth.

As in the cash $h > w^2$, one can show that $T(x)$ is a value function of the game with $h \leqslant w^2$.

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